MATH 2050C Lecture 14 (Mar 14)

Cauchy sequences (§ 3.5 in textbook)

Q: When is (x_n) convergent (without knowing its limit)?

A1: "MCT" bdd + monotone $\Rightarrow$ convergent

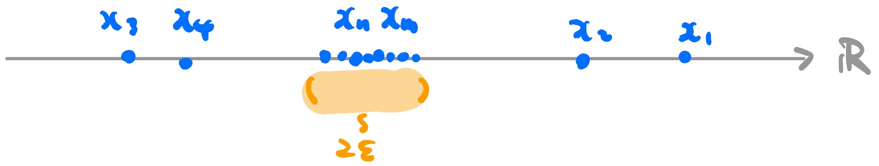
BUT "$\Leftarrow$" is FALSE E.g.) $(x_n) = \left(\frac{(-1)^n}{n}\right) \to 0$

A2: "Cauchy" $\Leftrightarrow$ convergent
"iff"

Def$^{\underline{n}}$: A seq. (x_n) is called Cauchy if

$\forall\, \varepsilon > 0,\ \exists\, H = H(\varepsilon) \in \mathbb{N}$ s.t.

$$|x_n - x_m| < \varepsilon \quad \forall\, n, m \geq H.$$



Remark: Compared to the ε-K def$^{\underline{n}}$ for convergence of (x_n), we DO NOT need to refer the potential limit x.

Example 1: $(x_n) := \left(\frac{1}{n}\right)$ is Cauchy. (Also $\left(\frac{1}{n}\right) \to 0$)

Pf: Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $H \in \mathbb{N}$ s.t. $H > \frac{2}{\varepsilon}$.

Then, $\forall\, n, m \geq H$,

$$|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{H} + \frac{1}{H} = \frac{2}{H} < \varepsilon$$

Example 2: $(x_n) := (1 + (-1)^n)$ is NOT Cauchy

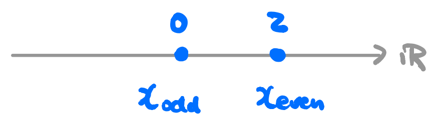
Pf: n odd : $x_n = 1 - 1 = 0$

n even : $x_n = 1 + 1 = 2$

$(x_n) = (0, 2, 0, 2, 0, 2, \dots)$

divergent!

Let $\varepsilon_0 = 1 > 0$. For any $H \in \mathbb{N}$ fixed.



$\exists$ odd $m \geq H$

$\exists$ even $n \geq H$

s.t. $|x_n - x_m| = |2 - 0| = 2 \geq 1 = \varepsilon_0$

$\blacksquare$

nec. & suff. condition

Thm: (x_n) convergent $\Longleftrightarrow$ (x_n) Cauchy

Proof: "$\Rightarrow$" Assume (x_n) is convergent, say $\lim(x_n) = x$.

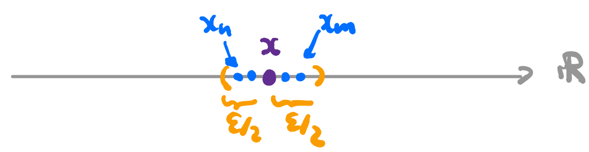
By def^n, let $\varepsilon > 0$ be given. then $\exists\ K = K(\frac{\varepsilon}{2}) \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon/2 \quad \forall n \geq K \qquad (*)$$

Choose $H = K \in \mathbb{N}$. Then $\forall n, m \geq H = K$,

$$|x_m - x_n| \leq |x_m - x| + |x_n - x| \overset{(*)}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So, (x_n) is Cauchy.



"$\Leftarrow$" Assume (x_n) is Cauchy.

Claim 1: (x_n) is bdd

Pf of Claim: Since (x_n) is Cauchy, take $\varepsilon_0 = 1 > 0$. then

$\exists\ H = H(1) \in \mathbb{N}$ s.t. $\forall n, m \geq H$,

$$|x_n - x_m| < 1 = \varepsilon_0$$

Fix $m = H$, then by reverse Δ-ineq and the above,

$$\big|\,|x_n| - |x_H|\,\big| \leq |x_n - x_H| < 1 \quad \forall\, n \geq H$$

$$\Rightarrow \qquad |x_n| \leq |x_H| + 1 \quad \forall\, n \geq H$$

Take $M := \max\{|x_1|, \dots, |x_{H-1}|, |x_H| + 1\}$

Then, $|x_n| \leq M$, $\forall\, n \in \mathbb{N}$. ie (x_n) is bdd.

Claim 2: (x_n) is convergent

Pf of Claim: Since (x_n) is bdd by Claim 1,

potential candidate for our limit

"BWT" $\Rightarrow \exists$ convergent subseq. $(x_{n_k}) \to x \in \mathbb{R}$.

Want to show: $(x_n) \to x$

By Cauchy def^{n}, let $\varepsilon > 0$ be fixed but arbitrary.

then $\exists\, H = H(\frac{\varepsilon}{2}) \in \mathbb{N}$ s.t.

$$|x_m - x_n| < \frac{\varepsilon}{2} \quad \forall\, n, m \geq H \quad \text{———} \ (*)$$

Since the subseq. $(x_{n_k}) \to x$ as $k \to \infty$, by def^{n}.

$\exists\, K = K(\frac{\varepsilon}{2}) \in \mathbb{N}$ st

$$|x_{n_k} - x| < \frac{\varepsilon}{2} \quad \forall\, k \geq K \quad \text{—} \ (**)$$

Fix a $k \geq K$ s.t $n_k \geq H$.

Then, $\forall\, n \geq H$, we have

$$|x_n - x| \leq \underbrace{|x_n - x_{n_k}|}_{(*)} + \underbrace{|x_{n_k} - x|}_{(**)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\square$

Example: Let (x_n) be the sequence defined by

$$x_1 := 1 \; ; \; x_2 := 2 \; ; \; x_n := \frac{1}{2}(x_{n-1} + x_{n-2}) \quad \forall\, n \geq 3.$$

Show that (x_n) is convergent and find $\lim (x_n)$.

Think: $(x_n) := (1, 2, 1.5, 1.75, 1.625, \ldots\ldots)$

bdd, NOT monotone, ……

Pf: By M.I. (Exercise), we have

- $1 \leq x_n \leq 2 \quad \forall\, n \in \mathbb{N}$
- $|x_{n+1} - x_n| = \dfrac{1}{2^{n-1}} \quad \forall\, n \in \mathbb{N}$

Claim: (x_n) is "Cauchy"

Pf of Claim: Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $H \in \mathbb{N}$ s.t. $H > \dfrac{4}{\varepsilon}$.

Then, $\forall\, m, n \geq H$, we want to show

$$|x_m - x_n| < \varepsilon \qquad \forall\, m, n \geq H$$

W.L.O.G. assume $m > n \geq H$.

$$|x_m - x_n| \leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$

$$= \frac{1}{2^{m-2}} + \frac{1}{2^{m-3}} + \cdots + \frac{1}{2^{n-1}}$$

$$= \frac{1}{2^{n-1}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-n-1}}\right)$$

$$< \frac{1}{2^{n-1}} \cdot 2 = \frac{1}{2^{n-2}} \leq 4 \cdot \frac{1}{2^H} \leq 4 \cdot \frac{1}{H} < \varepsilon$$

By Cauchy Criteria, $\lim (x_n) =: x$ exists.

Consider the subseq. $(x_{2k-1})_{k\in\mathbb{N}}$

<u>Note</u>: $\lim\limits_{k\to\infty} (x_{2k-1}) = x$

$$x_{2k-1} = 1 + \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots + \frac{1}{2^{2k-3}}$$

$$= 1 + \frac{\frac{1}{2}\left(1 - \frac{1}{4^{k-1}}\right)}{1 - \frac{1}{4}}$$

Take $k\to\infty$. we have $x = 1 + \dfrac{1/2}{3/4} = \dfrac{5}{3}$ #

$x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$

Take $n\to\infty$,

$x = \frac{1}{2}(x + x) = x$

not helpful